On Finite Time Intertemporal Dominance

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Abstract

We investigate intertemporal dominance conditions of finite unidimensional streams of outcomes in discrete time. We follow the time dominance approach based on unanimous net present value comparisons for classes of discounting factors representing temporal preferences. We first show that the infinite order of time dominance is equivalent to robust dominance for all classical exponential discounting factors with positive discount rate. We then parametrically restrict the class of discounting factors, by imposing a limit on the decrease of the weight attached in the current evaluation, between the outcomes of two future adjacent periods. Our results overcome the problem of dictatorship of the present in intertemporal evaluations and provide a parametric dominance condition that makes explicit the policy maker's trade-off, between current and future periods.

Keywords : Time Dominance, Discounting, Orderings, Sustainability.

1 Introduction

Increasingly today we are being asked to evaluate public projects, environmental policies or investments activities whose effects will be spread out over a number of years. Prominent examples include, for instance, global climate change, radioactive waste disposal, loss of biodiversity, thinning of stratospheric ozone, groundwater pollution, minerals depletion. These analysis involve in many cases intergenerational choices or, in general, intertemporal evaluations for which the specification of appropriate discount structures is crucial.

The evaluation of long term projects has been a source of strong debate in the economic literature and in cost-benefit analysis, questioning mainly the use of the

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standard exponential discount function. The exponential discount rate is considered unsatisfactory mainly because the consequences in the distant future become negligible as the weight attached to their current evaluation becomes very small.

Recently, the notion of *time declining discount rates* has gained considerable support from three main pillars of the economic literature. Experimental evidence suggests the use of hyperbolic discounting [Laibson, 1997; Loewenstein and Prelec, 1992] or its discrete time approximation, the quasi-hyperbolic discounting [Phelps and Pollak 1962]. Uncertainties about the discount factors, that may motivate the use of the Gamma discounting [Weitzman, 1998]. While the concern for avoidance of tyranny of future or present evaluations have motivated alternative rules in the social choice literature [Chichilnisky, 1996; Li and Löfgren, 2000].

This work is centered on deriving intertemporal dominance conditions for ordering finite streams of unidimensional cardinal outcomes, measured for instance as cash flows, distributed in discrete time and considering different classes of discounting functions subject to various restrictions. The analysis relates to several strands of literature including the work by Foster and Mitra (2003) on ranking investment projects in terms of net present value irrespective of the choice of the discount rate; Ekern (1981) which focuses on deriving intertemporal evaluations that are robust to the use of different formulas and values for discounting future benefits and costs; Karcher et al. (1995), Trannoy (2006) and Muller and Trannoy (2012) that investigate multidimensional stochastic dominance conditions with possible applications to intertemporal evaluations, and to more general multivariate stochastic dominance conditions as in Denuit *et al.* (2010). The discrete time setting of the model also relates it to Fishburn and Lavalle (1995) work on stochastic-dominance relations for probability distributions on a finite grid of evenly-spaced points and De La Cal and Cárcamo (2010) which present an analogous counterpart for inverse stochastic dominance conditions.

Moving from the time dominance work of Bøhren and Hansen (1980) and Ekern (1981) we extend existing results in two directions thereby investigating the possibility of increasing the comparability induced by the dominance conditions.

We analyze the impact of considering complete monotone discounting factors on the infinite order time dominance, and we further consider dominance conditions that overcome the issue of dictatorship of the present by restricting the class of discounting factors, imposing a limit on the decrease of the weight attached in the current evaluation between the outcomes of two future adjacent periods. Our main result will be implicitly characterized by the choice of two time thresholds: first, the horizon time T, that identifies the period after which the differences in the streams of cash flows are assumed to be negligible; second, the time period H, before which the discount structure gives enough weight to the outcomes, not allowing them to become negligible.

The paper proceeds as follows. In the next section we present in more details the different attitudes towards discounting and we discuss the notion of time dominance, presenting some preliminary tools and findings, necessary to achieve our main results in Section 3. In order to obtain a novel toolkit for evaluating projects we investigate the potential of the time dominance approach following two alternative routes. The first subsection of the Section 3 investigates infinite order time dominance conditions and the second one, parametrically restricts the set of discounting factors. Section 4 provides some concluding remarks.

2 Setting

Our approach is focusing on models of intertemporal comparisons having as a central point the discounting functions defined over streams of outcomes that are spread over a discrete and finite time span. While the choice of *discrete time* is driven mainly by the fact that empirical intertemporal comparisons of costs and benefit are commonly made over time grids, the choice of *finite time* is taken in order to avoid that the long future cash flows cumulate in an infinite advantage, giving in this way a dictatorship power of the future over the present.

We build on the setting introduced by Bøhren and Hansen (1980) and Ekern (1981), that derives stochastic dominance conditions applied to outcomes ordered on the time dimension. The decision maker evaluates intertemporal prospects by ranking streams of unidimensional cardinal outcomes, [cash flows, costs and benefits or utilities] arising from alternative projects in terms of higher *Net Present Value* (NPV). The ranking is made robust to all discount functions drawn from a particular class, which is defined by adding curvature restrictions on their derivatives with respect to time.

Consider two temporal profiles a and b represented by their return vectors $a = (a_0, a_1, ..., a_T) \in \mathbb{R}^{T+1}$ and $b = (b_0, b_1, ..., b_T) \in \mathbb{R}^{T+1}$, where time is discrete and finite and $t = 0$ denotes the present, while T is the horizon of the most long-lived project. For simplicity of notation, all projects are represented as having the same finite horizon T; if the projects had different horizons, the shorter vector could be augmented with zero entries to reconcile the size difference. When comparing them, project a will be strictly preferred to project b if and only it will lead to a higher NPV. The problem can be reformulated by considering the net project x

$$x = (x_0, x_1, ..., x_T) = (a - b) = (a_0 - b_0, ..., a_T - b_T)$$

and requiring its sign to be (strictly) positive.

In discrete time, for a given set of discount factors v, the NPV of x, written $NPV_v(x)$, is defined as

$$NPV_v(x) := \sum_{t=0}^{T} v_t \cdot x_t,$$

where the set $v = (v_0, v_1, \ldots, v_T)$ represents the temporal preferences and w.l.o.g. v_t will be normalized such that $v_0 = 1$.

The NPV function can be thought of as a polynomial in terms of the discounting functions v_t, with coefficient values that represent the net cashflows x_t.

2.1 How to evaluate the distant future?

The most widely used discounting structure in economics is the exponential discounting introduced by Samuelson (1937) and characterized by a constant rate of discount r, with

$$v_t = \delta^t, \text{where } \delta = \frac{1}{1+r}, \ \ r > 0. \tag{1}$$

A growing body of literature argues that, this conventional exponential discount factor turns out to be unsatisfactory when assessing sustainable development mainly because of its tendency to favor a myopic judgement of policies and, in the same time, for its inappropriateness to treat intergenerational issues [see, for example Schelling, 1995; Lind, 1995]. Recently, time declining discount rate has gained considerable support from theoretical and empirical studies. Table 1 presents the main discounting structures. First, experimental evidence suggests that people are more sensitive to a given time delay if it occurs closer to the present than if it occurs farther in the future, employing a higher discount rate to trade-offs now than to trade-offs in the future. This phenomenon is captured by the *hyperbolic discount factor* [Laibson, 1997; Loewenstein and Prelec, 1992].

Table1

Discount Factor	Expression
Hyperbolic	$v_t = (1+\gamma t)^{-\alpha/\gamma}, \ \alpha, \gamma > 0$
Proportional	$v_t = (1+\gamma t)^{-1}, \gamma > 0$
Power	$v_t = (1+t)^{-\alpha}, \alpha > 0$
Quasi-hyperbolic	$v_t = \beta(1+\delta)^{-t}, \ 0 < \beta \leq 1, \delta < 0$
Constant Sensitivity	$v_t = \exp[-(at)^b], \ a, b > 0.$

The parameter γ measures the extent of departure from the exponential function such that, if $\gamma \to 0$, v_t approaches the standard exponential function; if $\gamma \to \infty$ (very large), v_t approaches a step function; when $\gamma > 0$, v_t lies below the exponential function for low values of t and above it for high values of t. Given that α andγ are positive, the discount rates implied by the hyperbolic discounting decrease over time.

Proportional discounting [Hernstein, 1981] and *power discounting* [Harvey, 1986] are the special cases of hyperbolic discounting in which $\alpha = \gamma$ and $\gamma = 1$, respectively. The *quasi-hyperbolic discounting* was introduced Phelps and Pollak (1962) as a discrete approximation of the hyperbolic discounting, and the parameter β reflects the special status of the first period. Therefore, if $\beta = 1$ quasi hyperbolic discounting reduces to constant discounting, and if $0 < \beta < 1$ the discount structure mimics the qualitative properties of the hyperbolic function (higher today that tomorrow), while maintaining most of the analytical tractability of the exponential discount function (constant from tomorrow on). None of the above models is able to accommodate increasing impatience and this fact represents a drawback, in particular for individual analysis. Even though decreasing impatience is the common pattern (although there are studies that observe increasing impatience even at an aggregate level) there will always be individuals who are increasingly impatient. The *constant sensitivity model* [Ebert and Prelec, 2007], tractable as a hyperbolic model, can accommodate both moderately decreasing and increasing impatience. The parameter a in constant sensitivity reflects impatience and the parameter b the sensitivity to time. In other words it reflects the degree of decreasing impatience. For $b < 1$ a decision maker is decreasingly impatient, for $b > 1$ he is increasingly impatient, and for $b = 1$ constant sensitivity reduces to constant discounting.

A second pillar of literature, supporting time declining discount rate, is based on the work of Weitzman (1998) in which the clue lies in how one treats the uncertainty about the future. The uncertainty regarding the determinants of the discount rate, when modeled with a probability distribution (any), will justify a declining discount rate, moreover an hyperbolic shaped discount function. Therefore assuming a Gamma distribution with parameters μ, σ Weitzman obtained the *Gamma discount rate*

$$R(t) = \frac{\mu}{1 + t\sigma^2/\mu}.$$

Finally, the social choice literature simply says that a 'tyranny of the present' is not acceptable and that the discount rate issue should be determined by specific axioms that make such tyranny impossible. The contributions of Chichilnisky (1996) and Li and Löfgren (2000), even if different in approach, show that if one insists that there must be no dictatorship of one generation over another, the resulting program will have a discount rate which is a declining function of time.

In the next section we will discuss the notion of time dominance and show that all these attitudes towards discounting can be incorporated in it.

2.2 Aspects of Time Dominance

The time dominance (TD) approach applies the stochastic dominance methodology to a temporal context, where the consequences of a decision alternative are distributed over time. Considering that stochastic dominance puts successively stronger restrictions on the utility function representing risk preferences, TD restricts, in a similar way, the discounting functions representing temporal preferences. In this way the time dominance approach provides rules for a partial ordering of temporal prospects.

Therefore, TD calls for curvature restrictions to classify discounting functions, in analogy with what is done in the stochastic dominance literature, where assumptions are imposed on the utility functions, or on probability distortion functions. Let $v_t^0 = v_t$, then for any number $k = 1, 2, ..., n$ of restrictions imposed on the discount function, let

$$v_t^k = v_{t+1}^{k-1} - v_t^{k-1}.$$

Thus, v_t^k is obtained by differencing k times the function v_t. The widest class of discounting functions obtained for $k = 0$ and denoted V_0, requires simply that, at any point in time more is preferred to less. Formally,

$$V_0 = \{v : v_t > 0, \forall t \text{ with } v_0 = 1\}.$$

Downward sloping discounting functions representing time impatience belong to V_1, implying that a dollar at time t is preferred to a dollar at time $t + \Delta$ ($\Delta > 0$). The set V_2 contains the functions that are decreasing and convex in t [i.e. in the discrete case, with non increasing differences].

By adding successive restrictions on v_t^k, subsets of discounting functions are recursively defined:

$$V_k = \{v : v \in V_{k-1}, \text{ and } (-1)^k v_t^k > 0\}.$$

Hence, v belongs to the class V_n if and only if v_t^k alternates in sign (starting with a positive sign), as k goes from 0 to n. The domain of v_t^k is the set $\{0, 1, 2, \ldots, T-k\}$ in discrete time, in fact in discrete time, every time we add a condition on the discount functions v, we refer to a preceding period and therefore we loose a period from the finite set $\{0, 1, \ldots, T\}$. Following this logic, the number n of restrictions on the discounting functions can not overtake the horizon T.

Let $\mathbb{N}$ denote the set of natural numbers. We will also consider a larger family of sets V_k^* whose definition requires weak inequalities ($\geq$) to hold, clearly $V_k \subset V_k^*$ for all $k \in \mathbb{N}^* := \mathbb{N} \cup 0$. For this set of functions it is possible that $v_t = 0$ for some $t > 0$.

Definition 1 *$NPV_v(a) > NPV_v(b)$ for all $v \in V_n$ is denoted as $a \succ_n b$.*
$NPV_v(a) \geq NPV_v(b)$ for all $v \in V_n^*$ is denoted as $a \succcurlyeq_n^* b$.

The class of time dominance stochastic orders $\succ_n$ for $n \in \mathbb{N}$ has been investigated in Ekern (1981), the results for $\succcurlyeq_n^*$ for $n \in \mathbb{N}$ can be derived analogously.

Similar to the inverse stochastic dominance approach, TD concentrates on repeated summations of outcomes[1], resulting that the ordering of alternatives will depend on the mathematical properties of the cash flow distribution of net x.

Rewriting the initial cash flow $x_t = X_t^0$ and calling it Stage 0, using repeated summations, for $n = 1, 2, \ldots, T$, we get

$$X_t^n := \sum_{s=0}^{t} X_s^{n-1}.$$

[1]The main difference is that, while in the case of Inverse Stochastic Dominance, the outcomes are ranked according to their magnitute, in the TD the outcomes are ordered according to the time dimension.

The Stage 1 of repeated summations at any point in time is nothing but the sum of the initial cash flows, starting from 0 up to that point in time. Recursively, the Stage n of repeated summations at any point in time is sum of the previous level of repeated summations, up to that point in time, starting from 0.

Such repeated cumulations of consequences correspond to repeated cumulations of probabilities in the stochastic dominance approach, although there are some important differences. Unlike probabilities, cash flows may be negative and different projects' cash balances at the horizon do no necessarily coincide.

Hence X_t^k may decrease in t, and $X_a^1(T) \neq X_b^1(T)$. In fact, the TD conditions are more closely related with the inverse stochastic dominance conditions [see De La Cal and Cárcamo, 2010; Muliere and Scarsini, 1989; Yaari, 1987].

Using a time dominance methodology, the information about the X_t^k values for $1 \leq k \leq n$ may suffice to conclude whether some decision alternatives are definitely inferior for all discounting functions in the class V_n.

Definition 2 (TD of nthorder.) *i)Project a dominates b by the n^{th} order TD, denoted by $a \geqslant_n b$, if and only if for the net project $x = a - b$*

$$\begin{aligned} X_T^k &\geq 0 \textit{ for all } k = 1, 2, ..., n-1 \\ X_t^n &\geq 0 \textit{ for all } t \in \{0, 1, ..., T\}. \end{aligned}$$

*ii) Project a **strictly** dominates b by the n^{th} order TD, denoted by $a >_n b$, if there are strict inequality ($>$) holding for some comparisons.*

In order to illustrate the concept, consider the matrix $\{X_t^k\}$ presented in Table 2 where $k = 1, 2, ..., n$ and $t = 0, 1, ..., T$.

Table 2

Year	0	1	$\cdots$	t	$\cdots$	$T-1$	T
X_t^0	x_0	x_1	$\cdots$	x_t	$\cdots$	x_{T-1}	x_T
X_t^1	$\sum_{s=0}^{0} X_s^0$	$\sum_{s=0}^{1} X_s^0$	$\cdots$	$\sum_{s=0}^{t} X_s^0$	$\cdots$	$\sum_{s=0}^{T-1} X_s^0$	$\sum_{s=0}^{T} \mathbf{X}_s^0$
X_t^2	$\sum_{s=0}^{0} X^1(s)$	$\sum_{s=0}^{1} X^1(s)$		$\sum_{s=0}^{t} X_s^1$		$\sum_{s=0}^{T-1} \mathbf{X}_s^1$	$\sum_{s=0}^{T} X_s^1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$			$\vdots$
X_t^{k-1}	$\sum_{s=0}^{0} X_s^{k-2}$	$\sum_{s=0}^{1} X_s^{k-2}$		$\sum_{s=0}^{t} \mathbf{X}_s^{k-2}$			$\sum_{s=0}^{T} X_s^{k-2}$
X_t^k	$\sum_{s=0}^{0} X_s^{k-1}$	$\sum_{s=0}^{1} X_s^{k-1}$	$\cdots$	$\sum_{s=0}^{t} X_s^{k-1}$	$\cdots$		$\sum_{s=0}^{T} X_s^{k-1}$
X_t^T	$\sum_{s=0}^{0} \mathbf{X}_s^{T-1}$	$\sum_{s=0}^{1} \mathbf{X}_s^{T-1}$	$\cdots$	$\sum_{s=0}^{t} X_s^{T-1}$	$\cdots$		$\sum_{s=0}^{T} X_s^{T-1}$
X_t^{T+1}	$\sum_{s=0}^{0} \mathbf{X}_s^{T}$						

Note first that

$$\sum_{s=0}^{0} X_s^{T-1} = \sum_{s=0}^{0} X_s^0 = x_0,$$

which means that the first column of the matrix consists of identical elements that are equal to the first net cash flow x_0. Secondly,

$$X_t^k = \sum_{s=0}^{t} X_s^{k-1} = \sum_{s=0}^{t-1} X_s^{k-1} + X_t^{k-1},$$

so that each term in the matrix is the sum of the one before it (on the same row) and the one above it (from the same column). Third, for a given stage k of cumulation, the matrix elements participating in the time dominance conditions include all the elements of row k along with the terminal elements of the higher positioned rows, $k-1, k-2, ..., 1$. All these elements have been highlighted in Table 1 in bold face.

Note also that, whatever the TD of order k, the number of the matrix elements to be examined as required by the TD conditions always remains $(T+1)$. Thus, no additional restrictions are imposed if for a fixed T we consider the class $V_{T+\theta}$, for $\theta = 1, 2, \ldots,$ instead of V_T.

Another important note is that the cumulative values at every stage can be written as a summation of the initial values using binomial coefficients, as follows

$$X_t^k = \sum_{s=0}^{t} X_t^{k-1} = \sum_{j=0}^{t} x_j \cdot C_{k-1}^{t+(k-1)-j}, \text{ where } C_k^T = \frac{T!}{k!(T-k)!}.$$

Thus, the matrix on Table 2 can be rewritten using the combinatorics coefficients. The Example 1 illustrates this set of observations.

Ekern (1981) result relates the n^{th} order time (strict) dominance to the NPV (strict) superiority for discount functions in V_n.

Theorem 1 (Ekern 1981) *$a \succ_n b$ if and only if $a >_n b$.*

The strict TD of project a over b, $a \succ_n b$, implies that the test of n^{th} order TD is verified always for the weak inequalities and just sometimes for the strict inequalities. A "weaker" version of the theorem can also be derived in analogy with Ekern (1981) result.[2]

Remark 1 *$a \succcurlyeq_n^* b$ if and only if $a \geqslant_n b$.*

Example 1 *Consider a net project x with time horizon $T = 4$ and $n = 2$. The 2^{nd} order time dominance conditions, visualized in the table below, are associated with the*

[2] We omit here the straightforward proof.

non negative sign of the bold faced elements.

t	0	1	2	3	4
x_t	x_0	x_1	x_2	x_3	x_4
X_t^1	x_0	$\sum_{j=0}^{t=1} x_j \cdot C_0^{1-j}$	$\sum_{j=0}^{t=2} x_j \cdot C_0^{2-j}$	$\sum_{j=0}^{t=3} x_j \cdot C_0^{3-j}$	$\sum_{j=0}^{t=4} \boldsymbol{x}_j \boldsymbol{\cdot} \boldsymbol{C}_0^{4-j}$
X_t^2	$\boldsymbol{x}_0$	$\sum_{j=0}^{t=1} \mathbf{x}_j \boldsymbol{\cdot} \boldsymbol{C}_1^{2-j}$	$\sum_{j=0}^{t=2} \mathbf{x}_j \boldsymbol{\cdot} \boldsymbol{C}_1^{3-j}$	$\sum_{j=0}^{t=3} \boldsymbol{x}_j \boldsymbol{\cdot} \boldsymbol{C}_1^{4-j}$	
X_t^3	x_0	$\sum_{j=0}^{t=1} x_j \cdot C_2^{3-j}$	$\sum_{j=0}^{t=2} x_j \cdot C_2^{4-j}$		
X_t^4	x_0	$\sum_{j=0}^{t=1} x_j \cdot C_3^{4-j}$			
X_t^5	x_0				

Building on the framework presented above, the conditions of TD of a net project can be rewritten making use of a matrix of coefficients that are time dependent. The shape of this matrix depends on the horizon T, considered for the net project, and on the number of restrictions n imposed for the discounting functions.

We have seen in Example 1 that when $n < T$, meaning that the number of conditions imposed on the discount functions is lower than the discrete time horizon, the elements representing the time dominance conditions are the ones from the row n, together with the terminal elements of the higher positioned rows, $n-1, n-2, \ldots, 1$.

When $n = T$, the time dominance conditions require to check the sign of the elements on the diagonal of the matrix. Reconsidering the example above, the conditions would be

$$\begin{cases} \sum_{j=0}^{t=4} x_j \cdot C_0^{4-j} \geq 0 \\ \sum_{j=0}^{t=3} x_j \cdot C_1^{4-j} \geq 0 \\ \sum_{j=0}^{t=2} x_j \cdot C_2^{4-j} \geq 0 \\ \sum_{j=0}^{t=1} x_j \cdot C_3^{4-j} \geq 0 \\ x_0 \geq 0 \end{cases},$$

with strict inequalities for some conditions. Written in matrix form, the above mentioned comparisons become

$$(x_0, x_1, x_2, x_3, x_4) \cdot \begin{bmatrix} C_0^4 & C_1^4 & C_2^4 & C_3^4 & C_4^4 \\ C_0^3 & C_1^3 & C_2^3 & C_3^3 & 0 \\ C_0^2 & C_1^2 & C_2^2 & 0 & 0 \\ C_0^1 & C_1^1 & 0 & 0 & 0 \\ C_0^0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

Therefore, in order to have dominance, one should check the sign of the elements of the vector obtained from the vector of net cash flows x multiplied by a square matrix, denoted Π_T, of dimensions $(T+1)\times(T+1)$ that is

$$(x_0, x_1, \ldots, x_T)\cdot\begin{bmatrix} C_0^T & C_1^T & \cdots & C_{T-1}^T & C_T^T \\ C_0^{T-1} & C_1^{T-1} & \cdots & C_{T-1}^{T-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ C_0^1 & C_1^1 & \cdots & 0 & 0 \\ C_0^0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

where the nonzero entries of Π_T form an inverted Pascal's triangle (a triangle array of the binomial coefficients).

For the cases in which the product conditions, $x\cdot\Pi_T \geq 0$, are not conclusive, the dominance results can be improved by postponing the horizon T to a "longer horizon" $T^{'} > T$. The initial net project x of length T can thus be transformed into an equivalent project $x^{T^{'}}$of length $T^{'} > T$ by appending $T^{'} - T$ zeroes to x. Of course, adding zero net cash flows from time T onward does not influence the real attractiveness of either projects, but for purely mathematical reasons it may increase the discriminatory power of the dominance approach. Other scholars made some related observation with respect to strengthening their results by postponing the time horizon of the projects taken into consideration: Foster and Mitra (2002) on obtaining necessary and sufficient conditions for unambiguous dominance; Pratt and Hammond (1979) on strengthening the bound on the number of internal rates of return.

Example 2 *Let consider the net project* $x = (1, -3, 2.5)$ *with time horizon* $T = 2$. *In this case no TD result could be obtained, as* $X_1^1 < 0$ *and* $X_2^0 = x_2 > 0$, $X_0^2 > 0$. *In contrast, by sufficiently postponing the horizon into the future to* $T' = 6$, *time dominance can be established*

t	0	1	2	3	4	5	6
x_t	1	-3	**2.5**	0	0	0	0
X_t^1	1	$\mathbf{-2}$	0.5	0.5	0.5	0.5	**0.5**
X_t^2	**1**	-1	-0.5	0	0.5	**1**	2
X_t^3	1	0	-0.5	-0.5	**0**	1	3
X_t^4	**1**	**1**	**0.5**	**0**	0	1	3

.

In this way a new matrix $\Pi_{T'}$ can be constructed and the product rule, reapplied. The product of the net cash flow vector x with this new matrix yields not only a *sufficient* dominance condition, but also a *necessary* condition.

Proposition 1 *Let* $n > T$, *and* $x^n \in \mathbb{R}^{n+1}$ *where* $x_t^n := a_t - b_t$ *for* $t \in \{0, 1, \ldots, T\}$ *and* $x_t^n = 0$ *for* $t \in \{T+1, \ldots, n\}$, *then* $a >_n b$ *if and only if there exists a matrix* Π_n *such that* $x^n \cdot \Pi_n \geq 0$ *(with some* $>$*).*

In this way we can allow for the number of restrictions put on the discount functions to exceed the value of T. This novel representation of TD conditions is an adjustment that will contribute to the derivation of subsequent results.

How far can one postpone the horizon and in this way increase the number of restrictions put on the discount functions in order to find dominance? Is postponing the horizon at infinity going to guarantee some dominance? Starting from the TD structure presented above, we examine its applicability to different discounting functions and then, after restricting the set of admissible functions, we extend our analysis to infinite order time dominance.

3 Results

3.1 V_∞ time dominance

Notice that all the functions presented in the Section 2.1 are positively valued when $t \geq 0$ and have successive differences v_t^k, that alternate in sign, as k goes from 0 to n, being in line with the conditions imposed on the discounting functions from the TD approach seen in the Section 2.2. Knowing that we can allow for n, the number of restrictions put on the discount functions, to exceed the value of the discrete time horizon T, the next step is to incorporate time evaluations where $n \longrightarrow \infty$.

Considering V_n, the set of real valued functions v on $[0, \infty)$ whose recursive differences through order n alternate in signs, we can introduce an enlarged class of discount functions, V_∞.

Definition 3 *$V_\infty := V_n$ for $n \longrightarrow \infty$.*

The relation between finite and infinite degree TD concerns the following property of sequences of relations.

Lemma 1 (Thistle 1993) *Let $\{y_i\}^\infty$ be an infinite sequence, and let $\{y_i\}^n$ be the subsequence of the first n terms.*

If $\{y_i\}^n$ has property P for all finite n, then $\{y_i\}^\infty$ has property P.

We define the *Infinite TD* $a \succ_\infty b$ based on an approach by Thistle (1993). This result is of interest because it appears to be the only case where a ranking at a given degree of TD implies ranking at a lower degree.

Remark 2 (Thistle 1993) *$a \succ_\infty b$ if and only if $\exists\ n \in \mathbb{N}^*$ s. t. $a \succ_n b$.*

Proof. *Sufficiency.* Assume that $a \succ_\infty b$. Let us suppose that $a \not\succ_n b$. for all finite n. Applying the Lemma 1 this implies that $a \not\succ_\infty b$, contradicting the hypothesis. Therefore $a \succ_\infty b \longrightarrow a \succ_n b$ for some finite n.

Necessity. $a \succ_n b$ implies that $a \succ_{n+i} b$ for all $i \geq 1$. Thus $a \succ_n b \longrightarrow a \succ_\infty b$. ■

Note that all the discounting functions listed in Table 1 in Section 2.1 as well as the standard exponential discounting satisfy the conditions for being in V_n, and possess an additional property, which transposes in discrete time the notion of complete monotonicity[3]. In particular, the notion demands that the successive differences of the discount function v_t alternate in sign. By letting $n \longrightarrow \infty$ we can first remark that all these functions are in $V_{\infty,}$ and secondly, based on an approach introduced by Thistle (1993) on infinite stochastic dominance, we can relate finite time dominance with infinite one and denote it by $a >_\infty b$.

The assumptions made until now, may seem reasonable but next theorem is going to highlight some limitations of using them to rank intertemporal projects. In order to derive our result we first define the δ *Time Dominance*. We call δ *TD* the conditions (necessary and sufficient) for which the NPV of a is larger than the NPV of b (or equivalently the NPV of the net project x is positive) for all discount rates $r > 0$, or for all discount factors $\delta \in (0, 1)$, where $\delta = \frac{1}{1+r}$ and $r > 0$ with

$$NPV_\delta(x) = \sum_{t=0}^{T} \delta^t \cdot x_t.$$

Definition 4 *$NPV_\delta(a) > NPV_\delta(b)$ for every constant discount function $\delta \in (0, 1)$ is denoted as $a \succ_{\delta(0,1)} b$.*

The δ *TD* conditions have been investigated in Foster and Mitra (2002) and Karcher et al. (1995). By combining their results with what obtained earlier in Proposition 1 we get the following novel result.

Theorem 2 *$a \succ_\infty b$ if and only if $a \succ_{\delta(0,1)} b$.*

Proof. The *sufficiency* of the theorem results from $a \succ_\infty b$, that implies $\exists\, n$ s.t. $a \succ_n b$ for all $v \in V_n$. Note that the standard exponential function, $\delta(t) = \frac{1}{(1+r)^t}$, belongs to the set V_n for each $n \in \mathbb{N}^*$. It follows that $a \succ_\infty b \longrightarrow a \succ_{\delta(0,1)} b$.

For the *necessity* part, first we assume that $a \succ_{\delta(0,1)} b$. Then, the δ *TD* implies that there $\exists\, \overline{n}$ such that $x^{\overline{n}} \cdot \Pi_{\overline{n}} > 0$ (see Theorem 5, Foster and Mitra, 2002), and applying Proposition 1 we have that there exists a dominance of order $\overline{n}$, hence $a \succ_{\overline{n}} b$. Finally, subject to Remark 2, $a \succ_{\overline{n}} b \longrightarrow a \succ_\infty b$. ■

Thus, robust time dominance ranking for all functions in V_∞ coincides with dominance for all constant discount rates.

It is clear that $a \succ_\infty b$ implies $a \succ_{\delta(0,1)} b$ because all constant discount rate NPV are in V_∞, the relevance of the result is that this latter set of discount functions is also sufficient to get dominance conditions that holds for all functions in V_∞. This result is analogous to the one that highlights for continuous variables the equivalence

[3]In general, a functione $f(x)$ is completely monotone if it is positive valued and has derivatives that alternate in sign with $f' < 0$, $f'' > 0$, and so on.

between infinite stochastic dominance, completely monotone utility functions and dominance in terms of Laplace transforms [Fishburn and Willig, 1984; Thistle, 1993 and Shaked and Shantikumar, 2006, Ch. 5].

We have used the restricted set V_∞ as an alternative to the classic, constant discount rate, considering also the time declining discount rates, but this does not appear an adequate way of approaching the problem of ranking intertemporal projects. This is so for at least two limitations. First, the theorem states that keeping this wide subset of discount functions and requiring dominance for all of them, the ordering result obtained does not add anything more to the ordering results obtained by using just the conventional exponential function δ. Second, the present yields a dictatorship over the future, so that policies that give benefits for the generations in the distant future at the cost of those in the present are likely to be discarded even if benefits are substantial.

The first result deserves some comments that are presented in next subsection, while a possible solution to the second aspects will be derived in the following Section 3.2.

3.1.1 Aspects related to continuous time dominance

In order to parallel our result with those available for infinite stochastic dominance and its equivalence with dominance for all the exponential discounting functions [or Laplace Transform Dominance as in Shaked and Shantikumar, 2006, Ch. 5] we will briefly transpose those results within our framework. As we will show, for comparisons of discrete time streams, as is in our case, the existing results that are valid for continuos time dominance cannot be applied directly.

As in Ekern (1981), the TD relations for continuous time comparisons are associated with the NPV criterion over continuous net streams $x(t) = a(t) - b(t)$ with time weight $v(t)$, thus

$$NPV_v^c(x) = \int_0^T v(t) \cdot x(t)dt, \tag{2}$$

where the superscript c clarifies that the evaluation is made on continuous time. The repeated cumulations of the net cash flows in discrete time are then replaced with repeated integrations

$$X^n(t) := \int_0^t X^{n-1}(s)ds,$$

where $X^0(t) := x(t)$, while the curvature restrictions for classifying the discounting functions are represented by continuous derivatives that alternate in sign. In this way, the successive differences,observed in discrete time, mimic the local derivatives properties in the large. The subset V_n^c is obtained in the same way, by adding successive restrictions on $v^k(t)$, obtained, in this case, by differentiating $v(t)$, k times, i.e.

$$v^k(t) = \frac{dv^{k-1}(t)}{dt}.$$

Let us denote with V_n^c the set of discounting functions defined in continuous time, here we will focus on strict dominance conditions by replicating the definitions presented for discrete time evaluations.

Definition 5 *$NPV_v^c(a) > NPV_v^c(b)$ for all $v \in V_n^c$ is denoted as $a \succ_n^c b$.*

While the continuous analogous of the n^{th} order stochastic dominance is the following.

Definition 6 *Project a strictly dominates b by the n^{th} order continuous TD, denoted by $a >_n^c b$, if and only if for the net project $x(t) = a(t) - b(t)$*

$$\begin{aligned} X^k(T) &\geq 0 \text{ for all } k = 1, 2, ..., n-1, \\ X^n(t) &\geq 0 \text{ for all } t \in [0, T]. \end{aligned}$$

with strict inequality ($>$) holding for some comparisons.

As shown in Ekern (1981) $a \succ_n^c b$ if and only if $a >_n^c b$ for all $n \in \mathbb{N}$.

The continuous analogous of the infinite TD is denoted $a \succ_\infty^c b$ and coincides with the fact that $\exists\ n \in \mathbb{N}$ s.t. $a \succ_n^c b$.

The continuous exponential TD can be written as $a \succ_s^c b$ if and only if

$$\int_0^T \exp(-st) \cdot a(t)dt > \int_0^T \exp(-st) \cdot b(t)dt \text{ for all } s > 0.$$

By readjusting results in Fishburn and Willig [1984, Lemma 1] and Shaked and Shantikumar (2006, Ch. 5) one can derive the result $a \succ_\infty^c b \Leftrightarrow a \succ_s^c b$, which is the continuous analogous of Theorem 2.

However, it has to be pointed out that it is not the case that discrete TD and continuous TD are equivalent. In fact, as shown in Fishburn and Lavalle (1995), above the second degree of dominance the stochastic dominance relations for continuous variables do not coincides with those for variables defined on a grid [i.e. on a finite set of evenly spaced points]. By supplementing Fishburn and Lavalle (1995) results with those in De La Cal and Cárcamo (2010) and exploiting analogies between TD and inverse stochastic dominance one can derive the following implications for comparisons between discrete and continuous time dominance.

Remark 3 *If $n = \{1, 2\}$ then $a \succ_n b \Longleftrightarrow a \succ_n^c b$.*
If $n \geq 3$, with $n \in \mathbb{N}$, then $a \succ_n b \Rightarrow a \succ_n^c b$.

The first and second degree discrete TD relations are identical with their continuous counterparts. The equivalence of the partial sums approach with the traditional one, based on iterates of integrals of cumulative distribution functions fails from the third degree beyond, and discrete TD conditions prove to be more demanding then those defined in continuous time. So in principle we cannot use the continuous time results to derive the infinite dominance condition in discrete time. If fact, one may expect that, in line with Remark 3, also $a \succ_{\infty}^{c} b$ is implied by $a \succ_{\infty} b$ and given that $a \succ_{s}^{c} b$ implies $a \succ_{\delta(0,1)} b$ [by construction], then one can expect that $a \succ_{\infty} b \rightarrow a \succ_{\infty}^{c} b \leftrightarrow a \succ_{s}^{c} b \rightarrow a \succ_{\delta(0,1)} b$, thus by transitivity $a \succ_{\infty} b \rightarrow a \succ_{\delta(0,1)} b$.

However, the result in Theorem 2 shows that this is not the case, the two dominance conditions are equivalent thereby showing also that $a \succ_{\infty} b \leftrightarrow a \succ_{\infty}^{c} b$.

3.2 V^{α} restricted time dominance

The result in Theorem 2 asks for a reconsideration of the type of restrictions to put on the discounting function. Moving away from the successive restrictions scenario we turn back to the first subclass of functions V_1^*, positively valued and decreasing in time, thus those showing time impatience, and we introduce a parametric restrictions in order to overcome the issue of *dictatorship of the present.*

To deal with these limitations, in line with Chichilnisky (1996), we formalize the notion of Non Dictatorship of the Present (NDP).

Dictatorship of the present occurs if irrespective of future net positive benefits $\omega_t = \omega > 0$ for any $t \in \{h+1, \ldots, T\}$ there exists at least an admissible discount functions $v \in V_1^*$ such that if for $h \in \{0, 1, ..., T-1\}$, $x_h < 0$ and $x_t = 0$ for all $t \in \{0, 1, \ldots, h-1\}$ then $NPV_v(x) < 0$ where $x = (0, 0, ..., x_h, \omega, \omega, \ldots, \omega)$. We consider a weaker version of this condition requiring that $h < H$ for some $H \in \{1, ..., T\}$, that is the time period H denotes an upper bound beyond which future outcomes can be considered irrelevant for the social evaluation.

Next axioms postulates that this is not the case: the *Non Dictatorship of the Present* axiom does not allow a negative sign of the first net outcome to restrain future benefits from having a word to say in the judgement of the policy as long as some of these benefits take place no later than in period H. For a set $V_H \subseteq V_1^*$ of discounting functions, that can be conditioned on the value of H, the social evaluation $NPV_v(x)$ for $v \in V_H \subseteq V_1^*$ satisfies NDP if the following condition holds:

Axiom 1 (Non Dictatorship of the Present (NDP)) *Let $H \in \{1, ..., T\}$. For $x_h < 0$, where $h \in \{1, 2, \ldots, H-1\}$, $x_t = 0$ for all $t \in \{0, 1, ..., h-1\}$ and for any $v \in V_H \subseteq V_1^*$ there exists $\omega > 0$ with $x_t = \omega$, for all $t \in \{h+1, \ldots, T\}$ s. t. $NPV_v(x) \geq 0$.*

The following is a direct implication of axiom NDP on the ranking induced by $NPV_v(x)$ for $v \in V \subseteq V_1^*$.

Remark 4 *$NPV_v(x)$ for $v \in V_H \subseteq V_1^*$ satisfies NDP if and only if there exists a sequence $\alpha_t \in [0, 1)$ for $t \in \{0, 1, ..., T\}$ s.t. $\Delta_t = v_t - v_{t+1} \leq \alpha_t$ with $\sum_{t=0}^{H-1} \alpha_t < 1$.*

Proof. According to NDP $v_h x_h + \left(\sum_{t=h+1}^{T} v_t\right) \omega \geq 0$ should hold for all $v \in V_H \subseteq V_1^*$. If $v_h > 0$, this is the case only if $\left(\sum_{t=h+1}^{T} v_t\right) > 0$ where $h < H$. Recall that

$V_H \subseteq V_1^*$ thus all v_t are non increasing. Even if $v_t = 0$ for all $t > H$ the condition of $v_H > 0$ turns out to be necessary, but it is not sufficient. This aspect can be verified because if $h = H - 1$, when $v_t = 0$ for all $t > H$ the NDP condition requires that $v_{H-1}x_{H-1} + v_H\omega \geq 0$, however, for given values $v_{H-1} > 0$, $x_{H-1} < 0$ and $\omega > 0$ then for any $v_H \in (0, -v_{H-1}x_{H-1}/\omega)$ we will get that $NPV_v(x) < 0$. Thus it is not sufficient to have a positive values of v_H but it has also to be bounded above a positive level.

In fact, for a given $H \in \{1, ..., T\}$ a necessary and sufficient condition for NDP to hold is that there exists a value $\beta_H > 0$ such that $v_H \geq \beta_H > 0$. Given that we consider $v \in V_H \subseteq V_1^*$ then there exists a sequence of values $\beta_t > 0$ with $\beta_t \geq \beta_{t+1}$ for $t \in \{0, 1, \ldots, H\}$ such that $v_t \geq \beta_t > 0$ for all t. Recalling that $v_0 = 1$, and that $\Delta_t = v_t - v_{t+1} \geq 0$ for all t, the condition can be rephrased as $\alpha_t \in [0, 1)$ for $t \in \{0, 1, ..., T\}$ s.t. $\Delta_t \leq \alpha_t$ and $\sum_{t=0}^{H-1} \alpha_t < 1$. ■

In the remark it is simply required that the range of $\Delta_t = v_t - v_{t+1}$ is limited in the interval $[0, \alpha_t]$, where $\alpha_t < 1$ for each t, no further conditions are required for the values of v_t except those required by the fact that $v \in V_1^*$. Note that the condition $\sum_{t=0}^{H-1} \alpha_t < 1$ implies that $v_H = 1 - \sum_{t=0}^{H-1} \Delta_t \geq 1 - \sum_{t=0}^{H-1} \alpha_t > 0$.

In what follows we consider dominance for NPV functions focusing on discounting functions exhibiting a positive time preference $v \in V_1^*$ that satisfy NDP. These functions are parameterized by a common $\alpha \in [0, 1)$. Given that a necessary condition for NDP is that $v_H > 0$ for all admissible sets of discounting functions, then the choice of α will implicitly allow to identify the more distant threshold time period H. In fact, it should be that if $\Delta_t = \alpha$ for all $t \in \{0, 1, \ldots, H - 1\}$ we get $v_H = 1 - \sum_{t=0}^{H-1} \Delta_t = 1 - H \cdot \alpha > 0$, thus implying that $1/\alpha > H$. It follows that given $\alpha \in [0, 1)$ we get that the largest admissible value for H is $H = Int(1/\alpha)$ if $1/\alpha \neq Int(1/\alpha)$ where the operator $Int(x)$ selects the integer component of x, otherwise $H = 1/\alpha - 1$.

Let $\alpha \in [0, 1)$, then

$$V_1^\alpha := \{v \in V_1^* \text{ s.t. } \Delta_t = v_t - v_{t+1} \leq \alpha\}.$$

The parameter α can be interpreted as a magnitude restrictions on the fall of the discount function.

Note that dominance for all $v \in V_1^\alpha$ is implied by dominance in terms of NPV for all $v \in V_1^*$ that satisfy NDP.

Accordingly the dominance condition is the following:

Definition 7 *$NPV_v(a) \geq NPV_v(b)$ for all $v \in V_1^\alpha$ is denoted as $a \succcurlyeq_1^\alpha b$.*

In order to derive the associated TD conditions we make use of the curve $G_{X^{1*}}(t)$. The curve is obtained through the application of a double process of cumulation. First, net benefits are cumulated across time using the exogenous order of time and obtaining the values X_t^1 for $t = \{0, 1, \ldots, T\}$. Then these values are censored at the value of X_T^1, thereby obtaining $X_t^{1*} := \min\{X_t^1, X_T^1\}$. At the second stage the values of X_t^{1*} are ranked in non decreasing order leading to the distribution $X_{[t]}^{1*}$.

To conclude, these values are then cumulated leading to

$$G_{X^{1*}}(t) := \sum_{\tau=0}^{t} X^{1*}_{[\tau]} \tag{3}$$

for $t = 0, 1, ..., T$. The function can be extended to any value of $t > T$, by adding $(t - T)$ terms $X^1_T = X^{1*}_{[T]}$, that is, by evaluating it over an expanded stream of net benefits obtained adding a sequence of 0's to all periods after T. Thus we obtain,

$$G_{X^{1*}}(t) := \sum_{\tau=0}^{T} X^{1*}_{[\tau]} + (t - T) \cdot X^1_T$$

for $t > T$.While in general the linear interpolation of the curve gives for $t^* = Int(\theta)$ the formula

$$G_{X^{1*}}(\theta) = (\theta - t^*) \cdot X^1_{t^*+1} + G_{X^{1*}}(t^*). \tag{4}$$

The result in next theorem clarifies that in order to get a non negative net present value $NPV_v(x) \geq 0$ for all $v \in V^\alpha_1$ the value $G_{X^{1*}}(\frac{1}{\alpha} - 1)$ must be non negative.

Theorem 3 *For $\alpha \in (0, 1)$ then $a \succcurlyeq^\alpha_1 b$ if and only if $G_{X^{1*}}\left(\frac{1}{\alpha} - 1\right) \geq 0$.*
If $\alpha = 0$ then $a \succcurlyeq^0_1 b$ if and only if $X^1_T \geq 0$.

Proof. Recall that $a >^\alpha_1 b$ requires that $NPV_v(x) = \sum_{t=0}^{T} v_t \cdot x_t \geq 0$ for all $v \in V^\alpha_1$. These discounting functions satisfy the conditions (i) $\Delta_t = v_t - v_{t+1} \geq 0$ for all $t \in \{0, 1, \ldots, T\}$ with $v_{T+1} = 0$, and (ii) $\Delta_t \leq \alpha$. Writing $v_T = \Delta_T$, $v_{T-1} = \Delta_T + \Delta_{T-1}$ and in general $v_t = \sum_{k=t}^{T} \Delta_k$ we can therefore rewrite

$$\begin{aligned} NPV_v(x) &= \sum_{t=0}^{T} \left[\sum_{k=t}^{T} \Delta_k\right] \cdot x_t \\ &= \Delta_T \cdot \sum_{t=0}^{T} x_t + \Delta_{T-1} \cdot \sum_{t=0}^{T-1} x_t + \ldots + \Delta_1 \cdot \sum_{t=0}^{1} x_t + \Delta_0 \cdot x_0. \end{aligned}$$

Denoting $X^1_T := \sum_{t=0}^{T} x_t$ and substituting, we obtain

$$\begin{aligned} NPV_v(x) &= \Delta_T \cdot X^1_T + \Delta_{T-1} \cdot X^1_{T-1} + \Delta_{T-2} \cdot X^1_{T-2} + \ldots + \Delta_0 \cdot X^1_0 \\ &= \sum_{t=0}^{T} \Delta_t \cdot X^1_t. \end{aligned}$$

Recall that we need to check that $NPV_v(x) = \sum_{t=0}^{T} \Delta_t \cdot X^1_t \geq 0$ for all $\Delta_t \geq 0$ such that

$$\begin{cases} \Delta_t \in [0, \alpha] \text{ for } t \in \{0, 1, \ldots, T - 1\} \\ v_0 = \Delta_T + \Delta_{T-1} + \Delta_{T-2} + \ldots + \Delta_1 + \Delta_0 = 1, \end{cases} \tag{5}$$

and that X^1_t is obtained cumulating net cashflows, thus can also be negative.

Looking at the expression developed for the NPV and considering the (5) we note that it is like having a probability distribution where the $\Delta's$ are the weights and the NPV has the shape of an expected value. What would be the worst case scenario in which the NPV would have a negative sign? Of course this can happen when the highest weights are associated to the "most" negative summation of cash flows. Keeping this idea in mind one might find helpful to rank the values of $X_t^1 s$ in a non decreasing order, more or less like the principle that stays behind the Inverse Cumulative Distribution Function.

Necessity part. Rank the values of X_t^1's in non decreasing order, leading to the distribution $X_{[t]}^1$ for $t \in \{0, 1, \dots, T\}$ with

$$X_{[0]}^1 \leq X_{[1]}^1 \leq X_{[2]}^1 \leq \dots \leq X_{[T]}^1. \tag{6}$$

Note that if $\Delta_t = 0$ for all $t \in \{0, 1, \dots, T-1\}$ then every period receives the same weight equal to 1, $v_0 = v_1 = \dots = v_T = 1$, it follows that the *necessary* conditions for having dominance is that the cumulated sum of the initial net cash flows to the horizon, must be non negative $X_T^1 \geq 0$. This necessary condition holds irrespective of the value of α for any set V_1^α.

Assigning positive weights to the cumulated sums with higher values than X_T^1 increases the value of the NPV. However such configurations would not lead to the worst case scenario for the NPV dominance to check for deriving necessary conditions. In fact, by extending the time horizon by a string of 0's of appropriate length, will coincide with integrating the sequence (6) with terms of values X_T^1 without affecting the NPV of the net project. Thus for the derivation of necessary conditions all values of X_t^1 larger than X_T^1 should not be considered. Henceforth, the next step is to censure all the reordered cumulative values at the value of X_T^1,thereby obtaining $X_{[t]}^{1*} = \min\{X_{[t]}^1, X_T^1\}$. In this way, in the sequence (6) the X_T^1 receives all the weight that remains from the value of 1 after deducting the weights of the $X_{[t]}^1$ located before it.

Now we need to check when the NPV of these censored, rank dependent cumulative cash flows is non negative. For this purpose, the reordered values are cumulated again, introducing the following function

$$G_{X^{1*}}(t) := \sum_{\tau=0}^{t} X_{[\tau]}^{1*}$$

for $t = 0, 1, \dots, T$.

We are interested in the configuration leading to the lowest possible NPV, thus the smallest $X_{[t]}^{1*}$'s in the sequence have to receive the highest possible weight α. These weights have to satisfy condition (5), therefore if the first $t^* + 1$ elements in the sequence receive the maximum weight α, the next one gets the remaining weight $[1 - \alpha(t^* + 1)]$.

Now, if $\alpha T \geq 1$, then by denoting with $t^* = Int(\theta)$, where $\theta = (\frac{1}{\alpha} - 1)$,with $t^* \in \mathbb{N}$

and the function

$$
\begin{aligned}
NPV_v(x) &= \alpha \cdot \left(X^{1*}_{[0]} + X^{1*}_{[1]} + \ldots + X^{1*}_{[t^*]}\right) + [1 - \alpha(t^* + 1)] \cdot X^{1*}_{[t^*+1]} \geq 0 \\
NPV_v(x) &= \alpha \cdot \left[X^{1*}_{[0]} + X^{1*}_{[1]} + \ldots + X^{1*}_{[t^*]} + (\frac{1}{\alpha} - t^* - 1) \cdot X^{1*}_{[t^*+1]}\right] \geq 0
\end{aligned}
$$

we get

$$NPV_v(x) = \alpha \cdot G_{X^{1*}}(\theta) \tag{7}$$

where

$$G_{X^{1*}}(\theta) = (\theta - t^*) \cdot X^1_{t^*+1} + G_{X^{1*}}(t^*).$$

If $\alpha \cdot T < 1$ then the *necessary* condition requires that

$$NPV_v(x) = \alpha \cdot \left(X^{1*}_{[0]} + X^{1*}_{[1]} + \ldots + X^{1*}_{[T-1]}\right) + (1 - \alpha T) \cdot X^{1*}_{[T]} \geq 0$$

where by construction $X^{1*}_{[T]} = X^1_{[T]}$. The condition can be rewritten by adding further elements $X^{1*}_{[T]}$ to the sequence of cumulated values, i.e. adding 0's to the original string of net values, thereby re-obtaining

$$NPV_v(x) = \alpha \cdot \left[X^{1*}_{[0]} + X^{1*}_{[1]} + \ldots + X^{1*}_{[t]^*} + (\frac{1}{\alpha} - t^* - 1) \cdot X^{1*}_{[t^*+1]}\right] \geq 0.$$

The necessity condition thus requires that $G_{X^{1*}}(\frac{1}{\alpha} - 1) \geq 0$.

Sufficiency part. If $\alpha \in (0, 1)$ then by construction the NPV in (7), is the lowest among all possible NPV for all $v \in V^{\alpha}_1$. For a given distribution of values of X^1_t for $t = 0, 1, \ldots, T$, any other admissible distribution of values of Δ_t won't decrease the NPV. It follows that if $G_{X^{1*}}(\frac{1}{\alpha} - 1) \geq 0$ then $NPV_v(x) \geq 0$ for all $v \in V^{\alpha}_1$.

If $\alpha = 0$ then $NPV_v(x) = X^1_T$. ■

It is worth mentioning that when $\alpha = 0$, that is when every period receives the same weight equal to 1, then $v_0 = v_1 = \cdots = v_T = 1$. Thus, having a non negative sign of the cumulative cash flows at the horizon T, that is $X^1_T \geq 0$, becomes a *necessary* condition in order for one to choose project a over b.

When $\alpha \to 1$, the period 0 will be the only one receiving importance since all the future periods starting from $t = \{1, 2, \ldots, T\}$ will be discounted at values tending to zero. Therefore, $G_{X^{1*}}(0) \geq 0$ becomes a *necessary* condition, implying that $X^{1*}_{[0]} \geq 0$. Remember that by definition we have $0 \leq X^{1*}_{[0]} \leq X^{1*}_t \leq X^1_t$ for all t, and therefore $X^{1*}_{[0]}$ ≥ 0, by construction, implies $X^1_t \geq 0$ for all t, condition also required by $a >_1 b$.

Furthermore, if two projects do not verify the conditions for presenting 1^{st} order TD, the cumulated cashflows at the horizon represents a necessary condition, indicating when we can find a 1^{st} order $\alpha - TD$. This criterion has a common flavor with the 1st order Almost Stochastic Dominance criterion, a relaxation of the Stochastic Dominance concept that considers parametric restrictions of the set of utility functions based on the maximal ratio between marginal utilities at two different realizations within the domain [see Leshno and Levy, 2002]. The $\alpha - TD$ criterion considers instead a restriction on the *absolute magnitude* of the "marginal" change of the discount function between two adjacent periods. We can focus on absolute magnitudes because we apply a natural normalization of the discounting function by setting $v_0 = 1$. Nevertheless, even if the two criteria follow a similar approach the set of restrictions considered are logically distinct.

The $G_{X^{1*}}(t)$ function, produces a "verification criterion" for dominance of prospect a over b, in the sense that after α is specified, if the value of the function in $(\frac{1}{\alpha} - 1)$ is non negative, then so is the net present value $NPV_v(x)$. The next example clarifies the construction of the criterion.

Example 3 *Consider the following problem, where $x_t = a_t - b_t$, is the net cash flow for all $t = \{0, 1, 2, \ldots, 8\}$*

t	0	1	2	3	4	5	6	7	8
x_t	*-1*	*-2*	*3*	*3*	*-2*	*4*	*3*	*-2*	*-4*
X^1_t	*-1*	*-3*	*0*	*3*	*1*	*5*	*8*	*6*	$\boxed{\mathbf{2}}$
$X^1_{[t]}$	*-3*	*-1*	*0*	*1*	*2*	*3*	*5*	*6*	*8*
$X^{1*}_{[t]}$	*-3*	*-1*	*0*	*1*	*2*	*2*	*2*	*2*	*2*
$G_{X^{1*}}(t)$	*-3*	*-4*	*-4*	*-3*	*-1*	*1*	*3*	*5*	*7*

note that $G_{X^{1*}}(4.5) = 0$ thus $NPV_v(x) \geq 0$ if and only if $\frac{1}{\alpha} - 1 \geq 4.5$ that is $\frac{1}{5.5} = 0.1818 \geq \alpha^*$. Starting from the cut-off point α^* for which the function G is positive, the $NPV_v(x)$ will also be positive. In order to have dominance for all the functions in V_1^* the fall from period to period in the discount functions cannot be higher than the upper bound $\alpha^* = 0.18$.

This is to say that whenever we are dealing with a α higher that the cut-off point α^*we cannot find dominance of a project a over another one b.

It is of immediate verification the fact that as α decreases the dominance condition becomes less demanding.

Remark 5 *Let $\alpha' < \alpha$ then $a \succcurlyeq_1^{\alpha} b \rightarrow a \succcurlyeq_1^{\alpha'} b$.*

Consider the following example, in order to illustrate Remark 5.

Example 4 *Let $T = 6$, and the net cash flows $x_t = a_t - b_t$, for $t = \{0, 1, 2, \ldots, 6\}$ be represented in the table below*

t	0	1	2	3	4	5	6
x_t	*-2*	*2*	*2*	*2*	*2*	*2*	*-9*
X^1_t	*-2*	*0*	*2*	*4*	*6*	*8*	$\boxed{\mathbf{-1}}$
$X^1_{[t]}$	*-2*	*-1*	*0*	*2*	*4*	*6*	*8*
$X^{1*}_{[t]}$	*-2*	*-1*	*-1*	*-1*	*-1*	*-1*	*-1*
$G_{X^{1*}}(t)$	*-2*	*-3*	*-4*	*-5*	*-6*	*-7*	*-8*

In this example, a will never $\succcurlyeq_1^\alpha$ *b given that the verification function* $G_{X^{1*}}(t)$ *will be always negative for every* $t = \{0, 1, 2, \ldots, T\}$. *On the other hand, for any* $\alpha \leq 0.04$, $b \succcurlyeq_1^\alpha a$.

The $\alpha - TD$ criterion will solve comparisons when 1st order TD cannot be applied, in particular the sign of the non-discounted sum of net outcomes X_T^1 will play a crucial role to obtain $\alpha - TD$ dominance.

Remark 6 *When there is no 1st order TD, i.e., when neither* $a >_1 b$ *nor* $b >_1 a$, *then*

$$\begin{array}{l} (i) \text{ if } X_T^1 \geq 0 \text{ then } \exists\alpha \text{ s.t. } a \succcurlyeq_1^\alpha b \\ (ii) \text{ if } X_T^1 \leq 0 \text{ then } \exists\alpha \text{ s.t. } b \succcurlyeq_1^\alpha a \end{array}.$$

An implication of the remark is that one can not arrive to a disagreement point according to the $\alpha - TD$ criterion, that is, it is not possible that there exists an α for which $a \succcurlyeq_1^\alpha b$, and another α' for which $b \succcurlyeq_1^{\alpha'} a$. This feature will prove to be particularly relevant in next discussion.

There are clear links between our results and the literature on internal rates of return (IRR), both of them returning a value for which the net project has zero present value. However, one must keep in mind several important distinctions. In the IRR case, a rate of discount is obtained, that can be used as a cut-off between the range of discount rates that select one project and the range of rates that select the other, meanwhile in the case of $\alpha - TD$ criterion we get a magnitude restrictions on the fall of the discounting function. The former criterion can result in multiple IRR (when the sign of the net cash flow changes more than once during the project's life) for which the NPV equals 0 and therefore, the ranking can be reversed when choosing among them, whereas as shown in Remark 6 the $\alpha - TD$ criterion gives an unique value for which one project dominates another. Finally, the parameter α can be found for a large class of discounting functions, in our case for all those non-negative and non-increasing with time, whereas the IRR is often computed only for exponential discount function.

4 Conclusions

Is the TD approach suitable when facing sustainable intertemporal evaluations?

The TD structure can be adapted in order to include more functions than the unsatisfactory exponential discounting in assessing long term projects, but at the same time, even when going beyond the classical discount structure, as shown in Theorem 2, the ordering results are equivalent to the ones obtained in the standard case. We argue that, without any further (parametric) restrictions to the class of discounting function, the TD framework continues to give prevalent weight to the present in ranking streams of cash flows in terms of higher NPV. This result called for a method to control the problem of the dictatorship of the present. We have suggested the $\alpha-$*restricted TD* for discounting functions that exhibit time impatience, where α can be interpreted as a magnitude restrictions on the fall of the intertemporal weight. This approach gives a verification criteria, the $G_{X^{1*}}(t)$ function, for evaluating the long run projects, a criteria that makes explicit the trade-off between current and future periods.

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